

PHASE SPACE WARPING

A Dynamical Systems Approach to Diagnostics and Prognostics

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Abstract The concept of *phase space warping* is used to develop a general method for damage evolution tracking and failure prediction. After outlining the basic theory and describing its algorithmic implementation, experimental results are presented for a nonlinear oscillator in which a crack propagates to complete fracture. Our method is shown to give real-time estimates of the current “damage” state, and remaining useful life is accurately predicted well in advance of actual failure.

Keywords: dynamical systems, diagnostics, prognostics, failure prediction, condition monitoring, phase space reconstruction

Introduction

Most previous work done in the field of machinery condition monitoring has focused on the development of robust discriminators of *impending* failures. The work described here, however, aims to move past such alarm-based diagnostics to the actual tracking of *incipient* damage, which is required for a true prognostic capability capable of giving continuously updated estimates of remaining life. We describe a new,

general framework for damage evolution tracking, diagnosis, and prognosis, and apply the resulting method experimentally.

From the point of view taken here, damage evolution takes place in hierarchical dynamical system (Cusumano and Chatterjee, 2000) of the form:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \boldsymbol{\mu}(\boldsymbol{\phi}), t), \quad (1a)$$

$$\dot{\boldsymbol{\phi}} = \epsilon \mathbf{g}(\mathbf{x}, \boldsymbol{\phi}), \quad (1b)$$

where: $\mathbf{x} \in U \subset \mathbb{R}^m$ is the fast dynamic variable (the directly observable state); $\boldsymbol{\phi} \in V \subset \mathbb{R}^n$ is the slow dynamic variable (the hidden damage state); the parameter vector $\boldsymbol{\mu} \in \mathbb{R}^k$ is a function of $\boldsymbol{\phi}$; t is time; and the rate constant $0 < \epsilon \ll 1$ defines the time scale separation between the fast dynamics and slow “drift”.

To study systems of the form of Eqs. (1) experimentally, the concept of *phase space warping* is introduced, which refers to the small distortions that occur in the fast subsystem’s vector field as a result of the underlying slowly evolving damage process. After summarizing the basic theory and describing its implementation in an algorithm, we show the results of applying the method to a system with evolving material damage.

In (Chelidze et al., 2002; Cusumano et al., 2002), the method has been applied to the study of system in which the potential energy is perturbed by a battery-powered electromagnet: “failure” of the system in that case corresponded to complete discharge of the battery. It was shown that the tracking metric output by the algorithm was related in a 1-1, approximately linear fashion to the scalar generalized damage variable, which in that case was the open circuit battery voltage. Here, we apply the method to a vibrating beam nonlinear oscillator in which a crack propagates to complete fracture. Again, the tracking metric is shown to provide a 1-1 relationship with an independent measurement of the damage.

Using empirical damage evolution models and recursive filtering, the tracking metric can be used to predict remaining useful life. This approach as been applied to the battery experiment in a forthcoming paper (Chelidze and Cusumano, 2003), and here we apply it to the fracture experiment. In both cases, one finds that accurate, real-time estimates of current damage state and time to failure can be made well in advance of actual failures.

1. Phase Space Warping

In Eqs. (1), we assume that U and V are compact subsets. The phase space of the entire system of Eqs. (1) is the Cartesian product $\mathcal{S} = U \times V \times T$, where T is the manifold of which t is an element. We also assume that T is itself a compact manifold such as, for example, a p -torus corresponding to the vector field \mathbf{f} in Eq. (1a) being p -quasiperiodic in time. The key issue is that the dynamics of Eq. (1) must take place in a region of the extended fast phase space $\mathcal{F} = U \times T$ that is diffeomorphic (via delay coordinate embedding) with a compact subset $W \subset \mathbb{R}^d$ where d is the embedding dimension.

Our goal is to use only experimental measurements of the fast variable \mathbf{x} to observe, track, and ultimately predict, the slow hidden variable ϕ . Letting the constant ϕ_R represent the “reference” or initial value of the damage variable (which occurred at time t_R), and defining the prediction time $t_p = t - t_0$, we refer to the solution $\mathbf{x} = \mathbf{X}(t_p, t_0, \mathbf{x}_0, \boldsymbol{\mu}(\phi_R); \epsilon)$ as the *reference model*. We then define the time t_p ahead *short-time reference model prediction error* (STRMP) starting at time t_0 as

$$\mathbf{e}_R(\phi_0; t_p, t_0, \mathbf{x}_0) = \mathbf{X}(t_p, t_0, \mathbf{x}_0, \boldsymbol{\mu}(\phi_0); \epsilon) - \mathbf{X}(t_p, t_0, \mathbf{x}_0, \boldsymbol{\mu}(\phi_R); \epsilon), \quad (2)$$

in which the arguments for \mathbf{e}_R indicate that we consider it to be a map $\mathbf{e}_R: V \rightarrow \mathbb{R}^m$ with parameters t_p , t_0 , and \mathbf{x}_0 . We would like to understand conditions under which \mathbf{e}_R will provide a *tracking function*, that is, a smooth, injective mapping (preferably linear) from V into \mathbb{R}^m .¹

Equation (2) can be related to the fast vector field \mathbf{f} , since taking the derivative with respect t_p gives

$$\dot{\mathbf{e}}_R = \mathbf{f}(t_0, \mathbf{x}_0, \boldsymbol{\mu}(\phi_0)) - \mathbf{f}(t_0, \mathbf{x}_0, \boldsymbol{\mu}(\phi_R)). \quad (3)$$

Thus, $\dot{\mathbf{e}}_R$ measures the rate at which distortions are occurring in the vector field \mathbf{f} due to changes in the slow variable. We refer to such distortions as *phase space warping*. However, since it is difficult to measure vector fields directly in experiments, we use the flow form, Eq. (2), which gives the total amount of phase space warping at any given point $(\mathbf{x}_0, t_0) \in \mathcal{F}$.

For a fixed prediction time, we expect the solutions to be smooth both with respect to initial conditions and with respect to parameters. In addition, we require that the prediction time be “short”, that is $t_p \ll 1/\epsilon$. Then for any given initial time t_0 , the short prediction time allows the fast subsystem of Eq. (1a) to be treated as “quasistatic”, i.e. as having

¹In other words, \mathbf{e}_R is a tracking function if every point in its range corresponds to a unique point $\phi_0 \in V$.

an approximately constant damage variable ϕ_0 . Thus, using regular perturbations we expand the terms in Eq. (2) in a power series about $\epsilon = 0$, as $\mathbf{X} = \mathbf{X}_0 + \epsilon\mathbf{X}_1 + \dots \equiv \mathbf{X}(t_p, t_0, \mathbf{x}_0, \boldsymbol{\mu}(\phi_i); 0) + O(\epsilon t_p)$ where $\phi_i = \phi_0$ or ϕ_R . Substitution into Eq. (2) gives

$$\mathbf{e}_R = \mathbf{X}(t_p, t_0, \mathbf{x}_0, \boldsymbol{\mu}(\phi_0); 0) - \mathbf{X}(t_p, t_0, \mathbf{x}_0, \boldsymbol{\mu}(\phi_R); 0) + O(\epsilon t_p). \quad (4)$$

We further expand the leading term in Eq. (4) in a Taylor series about $\phi = \phi_R$, which upon substitution into Eq. (2) gives the STRMP error as

$$\mathbf{e}_R = \frac{\partial \mathbf{X}}{\partial \boldsymbol{\mu}} \frac{\partial \boldsymbol{\mu}}{\partial \phi} (\phi_0 - \phi_R) + O(\|\phi_0 - \phi_R\|^2) + O(\epsilon t_p), \quad (5)$$

in which we have suppressed the fact that the derivative matrices are evaluated at $(\epsilon, \phi) = (0, \phi_R)$.

Equation (5) shows that \mathbf{e}_R is to leading order an affine transformation of the slowly drifting variable,

$$\mathbf{e}_R \approx \mathbf{C}(t_p, t_0, \mathbf{x}_0, \phi_R) \phi + \mathbf{c}(t_p, t_0, \mathbf{x}_0, \phi_R), \quad (6)$$

where $\mathbf{C} = \frac{\partial \mathbf{X}}{\partial \boldsymbol{\mu}} \frac{\partial \boldsymbol{\mu}}{\partial \phi}$ is $m \times n$, and $\mathbf{c} = -\mathbf{C} \phi_R$ is $m \times 1$. We have dropped the zero subscript from ϕ_0 since the above discussion is true for *any* future value of the slow variable. However, the zero subscript is still required on t and \mathbf{x} since the matrices of Eq. (6) will depend on the selected initial point in the extended fast-time phase space, $(\mathbf{x}_0, t_0) \in \mathcal{F}$, used to compute the STRMP error.

A necessary condition for the transformation of Eq. (6) to be a tracking function is that the matrix $\mathbf{C} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ have maximal rank. We refer to this requirement on \mathbf{C} as the condition for *linear observability* for the slow variable.² Thus, for sufficiently small ϵ and $\|\phi_0 - \phi_R\|$, we can conclude that the STRMP error allows us to unambiguously track the changes in the slow variable ϕ using only measurements of the fast variable \mathbf{x} . Furthermore, we see that under ideal circumstances this tracking function can be expected to be approximately linear.

Given that the values of t_p , t_0 , \mathbf{x}_0 and ϕ_R are fixed, we can take the time derivative of Eq. (6) to find that $\dot{\mathbf{e}}_R \approx \mathbf{C} \dot{\phi} = \epsilon \mathbf{C} \mathbf{g}(\mathbf{x}, \phi)$. In other words, the rate of change of the tracking function, which is the rate at which phase space warping is occurring is, to leading order, determined by the slow vector field.

²We remark in passing that even when linear observability fails, higher order observability may be possible.

2. Algorithmic Implementation

Due to space limitations, in this paper it is only possible to highlight the key ideas used in the current implementation of the algorithm: the reader is referred to previous work (Cusumano et al., 2002; Chelidze et al., 2002; Chelidze and Cusumano, 2003) for further details.

In experiments the tracking function provided by the STRMP error, Eq. (6), is difficult to apply directly because \mathbf{C} and \mathbf{c} depend on \mathbf{x}_0 , t_0 , and t_p , which can be thought of as parameters. Unfortunately, for most applications it is difficult or impossible to repeatedly start the fast subsystem Eq. (1a) from the same initial conditions (\mathbf{x}_0, t_0) . Thus one should use many values of $(\mathbf{x}_0, t_0) \in \mathcal{E} \subset \mathcal{F}$ from some ensemble \mathcal{E} to deal with initial state repeatability problem as well as to increase the robustness of the method. Then for every fixed ϕ , the STRMP tracking function \mathbf{e}_R can be thought of as a random variable determined by random maps of the form of Eq. (6), with each map corresponding to a single element of \mathcal{E} .

Given such an ensemble of tracking functions, one can examine the multivariate statistics of the random vector \mathbf{e}_R in order to obtain information about the structure of the slow phase space. This general approach is the subject of current research by the authors. For example, in its simplest form this might mean replacing the map of Eq. (6) by $\langle \mathbf{e}_R \rangle = \langle \mathbf{C} \rangle \phi + \langle \mathbf{c} \rangle$, where the angled brackets indicate the average over \mathcal{E} .

If it is reasonable to assume, given the physics of the problem, that the slow variable is both *monotonic in time* and *scalar*, then an even simpler approach is to consider only the magnitude of \mathbf{e}_R , which gives to leading order

$$\langle \|\mathbf{e}_R\| \rangle \equiv e_R = C\phi + c, \quad (7)$$

where $C = \langle \|\mathbf{C}\| \rangle$ and $c = -C\phi_R$. This is the approach taken in the work presented here.

2.1 Damage tracking

In practice, the existence of a tracking function is used experimentally *as a hypothesis*, since it is generally the case that measurements of the damage variable are unavailable (indeed, that is the entire motivation for the method). In general, then, under the assumptions presented above, the output of the tracking function will identify the damage state *to within an unknown, approximately affine transformation*. However, in cases where independent measurements of the damage variable are available the tracking function can, in effect, be calibrated, so that the exact transformation can be determined.

Some form of phase space reconstruction, for example using delay coordinate embedding (Takens, 1981; Sauer et al., 1991) for initially chaotic systems or stochastic interrogation (Cusumano and Kimble, 1995) for nonchaotic systems, can be used to generate \mathcal{E} . In the current implementation, we prepare our system to be chaotic in its reference condition, and data is collected in a sequence of time intervals of length $t_D \ll O(1/\epsilon)$. The required ensemble \mathcal{E} is then obtained by using all of the data in a given interval, with a fixed prediction time $t_p < t_D \ll O(1/\epsilon)$. The delay time τ and embedding dimension d are determined using the first minimum of the average mutual information (Fraser and Swinney, 1986) and the method of false nearest neighbors (Kennel et al., 1992), respectively.

Since the form of the governing differential equations is assumed to be unknown, we instead consider a map

$$\mathbf{y}(r+1) = \mathbf{P}(\mathbf{y}(r); \phi), \quad (8)$$

where $\mathbf{y}(r)$ is the reconstructed fast state at time step r . Then, the tracking function of Eq. (2) is

$$\mathbf{e}_R(\phi; k, r) = \mathbf{P}^k(\mathbf{y}(r), \phi) - \mathbf{P}^k(\mathbf{y}(r), \phi_R), \quad (9)$$

where \mathbf{P}^k is the k^{th} iterate of the map defined in Eq. (8).

The leading term on the right hand side of Eq. (9) is simply $\mathbf{y}(r+k)$, and is available from the data. There are various ways that one might estimate the reference model $\mathbf{P}(\cdot; \phi_R)$: in this work, we use locally linear models

$$\mathbf{y}(r+1) = \mathbf{A}(r)\mathbf{y}(r) + \mathbf{a}(r), \quad (10)$$

where $\mathbf{A}(r)$ is a $d \times d$ matrix and $\mathbf{a}(r)$ is a $d \times 1$ vector. The parameters of the local linear models are determined using regression on the N nearest neighbors of $\mathbf{y}(r)$ and their future states *for data taken in the reference condition*. Then the damage tracking function Eq. (9) can be written as

$$\mathbf{e}_R = \mathbf{y}(r+k) - \mathbf{A}^k \mathbf{y}(r) - \mathbf{a}^k + \mathbf{E}^M(r) = \mathbf{E}_k(r, \phi) + \mathbf{E}^M(r), \quad (11)$$

where, for simplicity, we have suppressed the dependency of \mathbf{A} and \mathbf{a} on r . In the above equation, $\mathbf{E}^M(r)$ represents the model error and

$$\mathbf{E}_k(r, \phi) = \mathbf{y}(r+k) - \mathbf{A}^k \mathbf{y}(r) - \mathbf{a}^k \quad (12)$$

is the *estimated tracking function* that can be determined experimentally.

We wish to use the ensemble-averaged scalar tracking function Eq. (7), which can be experimentally estimated as $e_R = \langle \|\mathbf{e}_R\| \rangle \approx \langle F(\|\mathbf{E}_k\|) \rangle$, where F is a suitable filter. The filter is needed because now there are two spurious fluctuations in e_R that occur as one moves from point to

point in the reconstructed fast phase space, both of which occur because we must estimate \mathbf{e}_R with \mathbf{E}_k . The first is due to changes the accuracy of the linear map Eq. (10), even in the absence of noise. The second is caused by experimental noise.³ The effect of both fluctuations can be significantly reduced (Chelidze and Cusumano, 2003) by taking F to be a simple Kalman filter. In the language of recursive filtering, the first of the above fluctuations can be treated as “process noise”, whereas the second is “measurement noise”.

2.2 Estimation of remaining life

Application of the ideas outlined in the previous section results in a *damage tracking time series* over the slow time scale. Given the form of the hierarchical system Eqs. (1), the dynamics of the slow variable will be closely approximated by the solution to the slow flow equation obtained by replacing the slow vector field g by its long time average (Cusumano and Chatterjee, 2000).⁴ This means that one can consider candidate damage models of the form

$$\dot{\phi} = \epsilon \bar{g}(\phi) , \quad (13)$$

where \bar{g} is the long-time average of $g(\mathbf{x}, \phi)$.⁵ Note that we are *not* assuming that averaging can be applied analytically: rather, we use the concept of averaging only to justify the *autonomous* form of Eq. (13). A suitable model structure must be found from first principles or empirically from prior applications of the tracking algorithm.

Given the damage model Eq. (13), the time to failure can be estimated from the tracking function output. Again, recursive estimation is used, but in this case the “process” is typically nonlinear, and so a nonlinear method, such as an extended Kalman filter, or unscented filtering (Julier and Uhlmann, 1997), must be used. The difference equations used to define the estimator treat the damage tracking time series as a series of *observations* from which one wishes to estimate the actual damage state using the “sensor model” of Eq. (7). Thus, a side benefit of using the recursive filter with a specific damage model is that one is able to obtain an estimate of the actual damage state consistent with the model.

³Note that the “model fit error” \mathbf{E}^M in Eq. (11) includes both of these sources of error.

⁴Since ϕ is considered to be a scalar, so is g .

⁵It is worth noting that the effect of “load”, i.e. the amplitude of \mathbf{x} , enters through its average effect on g .

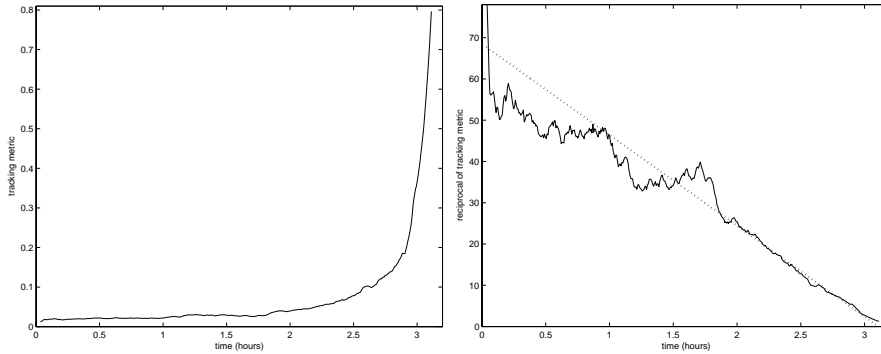


Figure 1. Damage tracking and failure prediction for a beam with growing crack: (*left*) plot of unscaled tracking metric vs. time showing simple power law behavior despite the complex dynamics of the beam; (*right*) estimated time to failure vs. time. Each black dot is the average one-step-prediction-error in a 2^{14} point record of fast time (100 Hz) strain gauge data. The gray line in the right figure is the actual time to failure, know *a posteriori*. The convergence to the actual time to failure occurs well before the actual failure.

3. Experimental application

For the experiments described here, we used the well-known two-well magneto-elastic oscillator, modified as described in (Chelidze et al., 2002). In the system, a clamped-free beam is restricted to a single degree of freedom by stiffeners. Two rare-earth magnets near its free end provide a two-well potential. The beam displacement is measured by a strain gauge mounted close to the clamped end. A shallow notch is machined in the beam below the strain gauge and just above the stiffeners. The system is mounted on a shaker and is forced at 8 Hz. The damage in the beam accumulates slowly and the experiment is run until complete fracture of the beam. Strain gauge output is sampled at 160 Hz sampling frequency, digitized (16 bit A/D), and stored on a computer.

The experiment was stopped after approximately every 10 minutes of data acquisition to take a digital image of the beam profile near the notch. During the image acquisition, the beam was always positioned in the same potential well, so that the surface of the notch was in tension. After taking the image, forcing was restarted and collection resumed after letting initial transients die out. As the experiment progressed, there was a decrease in the beam stiffness at the notch caused by fatigue damage accumulation. As a measure of this damage, we estimated the change in the slope of the beam across the notch using the acquired digital images.

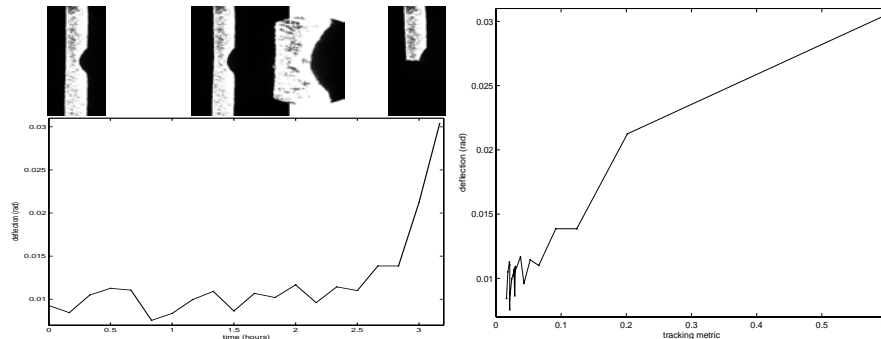


Figure 2. Calibration of tracking metric with independent measurement of damage state: (left, top) photos of the notch taken at 0, 2.5 and 3 hours, showing the first visible crack at 2.5 hours. (left, bottom) Plot of angular deflection across notch during experiment, calculated from image; (right) plot of tracking metric vs. angular deflection. The approximately linear, 1-1 relationship is consistent with the assumptions of the phase space warping approach described in the text.

Delay time and embedding dimension were estimated to be $6t_s$ and 5, respectively. The first 2^{14} data points were used for the reference data set, and $N = 16$ nearest neighbors were used for the local linear model parameter estimation. After going through the embedding and modeling process, we split data into 456 non-overlapping records of $t_D = 4 \times 10^3 t_s$ size. The tracking function \mathbf{S}_1 was estimated by calculating the short-time prediction error $\mathbf{E}_1(\mathbf{y})$ of the reference model for each record.

For time-to-failure estimation we used a power law model, $g = \phi^\alpha$, with the final value of ϕ taken to be $\phi_F = 0.645$. The values $\epsilon = 0.0061$ and $\alpha = 2$ were used in the recursive time-to-failure estimation procedure. The results for of this process are show in Fig. 1.

The damage tracking time series, in Fig. 1(left) shows smooth power law behavior, even though the actual load history at the notch is quite complex, consisting of many chaotic/periodic transitions. Figure 1(right) demonstrates that the time-to-failure estimate converges to the true value (known *a posteriori*) well in advance (about 1 hour out of 3 total) of the total failure of the beam. The time required for convergence is related to the fact that it is very difficult to carry out the required estimation when the tracking function is in its initial, almost flat condition prior to about the second hour of the experiment. The fact that the simple $\alpha = 2$ model works here is remarkable, and is consistent with Paris' law, even though the crack loading is decidedly not periodic.

In this case, we also have the independent measurement of the beam damage, which is shown in Fig. 2. In Fig. 2(left, top) we sample im-

ages taken during the experiment at (from left to right) 0, 2.5 and 3 hours. The first visible crack occurred at the 2.5 hour mark, but actual prediction of time to failure converged somewhat before that point, and tracking even earlier, indicating the sensitivity of the tracking method. Figure 2(left, bottom) shows the static angular deflection across the notch computed from the images. It is seen to increase in a way that is qualitatively similar to the tracking function output. Finally, Fig. 2(right) presents a plot of the angular deflection data vs. the tracking metric. We see an approximately linear, 1-1 relationship, consistent with the phase space warping theory presented in Section 1.

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